ON THE THEORY OF WAVES IN NONSTATIONARY COMPRESSIBLE MEDIA

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Motions of nonhomogeneous media which are close to characteristic (to Riemann waves) have been studied in detail in the literature on the propagation of waves with small wave length, i.e. in the approximation of geometric acoustics. These studies have included both the linear case [1-3] and consideration of nonlinear effects [4-7]. The results which have been obtained refer to media with properties which do not vary with time. Naturally, any variation of the parameters of the medium with time introduces a number of peculiarities into the process of propagation. In particular the duration (period) and the energy of a wave will vary even in the absence of any dissipative effects.

In this paper waves will be considered in media, the parameters of which depend on the coordinate x and the time t; i.e. the medium moves in any specified manner in the direction of the x-axis. The method used below is somewhat different from the usual methods of characteristics. In certain cases it permits reduction of the problem to successive integration of first-order partial differential equations.

The method is first applied to the problem in which the undisturbed motion (undisturbed by the wave under consideration) is a simple (Riemann) wave. Here it is possible to obtain the general solution describing acoustic disturbances of arbitrary shape. However, for arbitrary initial motion of the medium the solutions which are found and investigated generalize the approximation of geometric acoustics to the case of waves in nonstationary media, including waves of finite amplitude.

1. Interaction of acoustic and Riemann waves. In the absence of dissipative effects, the basic equations of gas dynamics have the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \qquad \frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0$$
$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}=0, \quad p=p(\rho, S), \qquad c^2=\left(\frac{\partial p}{\partial \rho}\right)_S \tag{1.1}$$

Here u is the velocity, ρ the density, p the pressure, c the speed of sound and S the entropy. For the present, external forces are considered to be absent.

Keeping in mind that the motion differs only slightly from a Riemann wave, we shall look for the derivative of one of the quantities (for example u) and the relation between the remaining quantities and u on a characteristic. Considering for definiteness a c_+-characteristic, we obtain

$$\left(\frac{du}{dt}\right)_+=\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=\varphi(x,t), \qquad u=u_0+\int\limits_{\rho_0}^{\rho}\frac{c}{\rho}\,d\rho+\psi(x,t) \qquad (v=u+c) \tag{1.2}$$

where φ and ψ are unknown functions. The substitution of (1.2) into (1.1) results in equations which are linear in φ and ψ for isentropic motion

$$\frac{\partial \psi}{\partial t}+(u-c)\frac{\partial \psi}{\partial x}=0, \qquad \varphi=c\,\frac{\partial \psi}{\partial x} \tag{1.3}$$

Therefore, ψ is propagated along a c_--characteristic.

In accordance with what has been stated above, let us set $u = u^\circ(\xi) + u'$, $\rho = \rho^\circ(\xi) + \rho'$, etc., where u', ρ' are small functions and u°, ρ° constitute a simple wave, i.e. satisfy (1.2) with $\varphi \equiv 0$, $\psi \equiv 0$ and depend on the variable $\xi = x - v^\circ t$. Then, in the first approximation we may set $u = u^\circ$, $c = c^\circ$ in (1.3) (since φ and ψ are small, only second-order terms are discarded by doing this). In order to integrate (1.3) we transform from x and t to the variables ξ, $t' = t$ and take account of the fact that

$$\frac{\partial \xi}{\partial x}=-\frac{1}{v^\circ}\frac{\partial \xi}{\partial t}=1-t'\frac{dv^\circ}{d\xi}\frac{\partial \xi}{\partial x}=\left(1+t'\frac{dv^\circ}{d\xi}\right)^{-1}$$

In these variables the equation of a c_--characteristic is linear in t'

$$\left(\frac{dt'}{d\xi}\right)_-=\frac{1}{2c}\left(1+t'\frac{dv^\circ}{d\xi}\right)$$

As a result we find the integral of (1.3)

$$\psi = F\left(t'\sqrt{c^\circ\rho^\circ} \mp \frac{1}{2}\int\limits_{\xi_0}^{\xi}\sqrt{\frac{\rho^\circ}{c^\circ}}\,d\xi\right), \qquad \varphi = \sqrt{c^\circ\rho^\circ}\,\frac{dF}{dy} \tag{1.4}$$

Here and later, F is an arbitrary function determined by the initial and boundary conditions, y is the argument of F.

Considering the second equation of (1.2) and (1.4), we represent the first equation of (1.2) in terms of the small disturbances u' in the form

$$\frac{\partial u'}{\partial t} + v^\circ\frac{\partial u'}{\partial x} + \frac{u'}{c^\circ}\frac{\partial c^\circ\rho^\circ}{\partial\rho^\circ}\frac{\partial u^\circ}{\partial x} = \frac{\rho^\circ}{c^\circ}\frac{\partial c^\circ}{\partial\rho^\circ}\frac{\partial u^\circ}{\partial x}\,F(y) + \sqrt{c^\circ\rho^\circ}\,\frac{dF}{dy} \tag{1.5}$$

After transformation to the variables ξ and t', equation (1.5) does not contain the derivative $\partial u'/\partial\xi$ and is immediately integrated

$$u' = \left(1 + t'\frac{dc^\circ}{d\xi}\right)^{-1}\left\{\Phi(\xi) + \frac{1}{2}\left[F\left(1 + t'\frac{dv^\circ}{d\xi}\right) - \frac{1}{\sqrt{c^\circ\rho^\circ}}\frac{dv^\circ}{d\xi}\int F\,dy\right]\right\} \tag{1.6}$$

Equation (1.6) is the general integral of the system (1.1) linearized about $u^\circ(\xi)$ and $\rho^\circ(\xi)$. As was to be expected, it contains two arbitrary functions, Φ and F. For constant u° and ρ° we have, of course, the usual result

$$u' = \Phi\left[x - (u^\circ + c^\circ)t\right] + F\left[x - (u^\circ - c^\circ)t\right]$$

Let us now set $F \equiv 0$ in (1.6); we then obtain a wave moving in the positive direction. As is apparent from (1.2), the functions $u = u^\circ + u'$, and $\rho = \rho^\circ + \rho'$ form a simple wave. $\Phi = 0$ corresponds to the more interesting case of a wave propagating in the opposite direction. For instance, let an acoustic wave propagate oppositely to a simple wave which connects the regions of constant flow (1) and (2). The acoustic wave propagates (in region (1)) with a frequency $\omega^{(1)}$ and amplitude u^m. Then, determining the $F(y)$ in (1.6), we find*

$$\frac{u'}{u^m} = \sin\left(\frac{2\omega^{(1)}\sqrt{c^{(1)}/\rho^{(1)}}}{c^{(1)} - u^{(1)}}\,y\right) + \frac{c^{(1)} - u^{(1)}}{2\omega^{(1)}c^{(1)}}\sqrt{\frac{c^{(1)}\rho^{(1)}}{c^\circ\rho^\circ}}\,\frac{dv^\circ/d\xi}{1 + t'\,dv/d\xi}\times$$
$$\times\cos\left(\frac{2\omega^{(1)}\sqrt{c^{(1)}/\rho^{(1)}}}{c^{(1)} - u^{(1)}}\,y\right) \tag{1.7}$$

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* We assume for simplicity that the derivative $dv^\circ/d\xi$ is everywhere continuous.

(the symbol $^{\circ}$ is omitted for constant quantities in region (1)). It follows from the above that after the wave passes into region (2), it will be sinusoidal with the same amplitude u^m (i.e. the amplitude of p' will vary proportionally to $c^{\circ}\rho^{\circ}$) and with the frequency $\omega^{(2)}$

$$\omega^{(2)} = \omega^{(1)} \frac{[(u^{\circ} - c^{\circ}) \sqrt{\rho^{\circ} / c^{\circ}}]^{(2)}}{[(u^{\circ} - c^{\circ}) \sqrt{\rho^{\circ}/c^{\circ}}]^{(1)}} \tag{1.8}$$

The case $u^{\circ} = c^{\circ}$ means simply that the wave is stationary in the given region (in the coordinate system chosen); i.e. u' does not depend upon t.

If there is a region of compression in the simple wave, then for $1 + t'dv^{\circ}/d\xi = 0$, which corresponds to the formation of a shock wave, the expressions (1.6) and (1.7) diverge*. The solution of the problem of interaction of an acoustic wave and a shock wave differs from (1.6) and is well known for a stationary discontinuity [1]. Equation (1.6) together with this solution permits determination of the result of passage of a sound wave through a wave of finite extent consisting of a discontinuity and a wave of rarefaction trailing behind it.

It is also easy to determine the behavior of an entropy wave in a region of Riemann motion. To do this we set $u = u^{\circ}$ and $S = S^{\circ} + S'$ in the third equation of (1.1) and again transform to the variables ξ and t'. As a result we obtain

$$S' = F_s \left(t'\rho^{\circ}c^{\circ} + \int_{\xi_0}^{\xi} \rho^{\circ}\, d\xi\right) \qquad (F_s \text{ is an arbitrary function}) \tag{1.9}$$

We remark also that if the frequency $\omega^{(1)}$ is sufficiently large ($dv^{\circ}/d\xi$ is small), the second term in (1.7) can be neglected. Then u' depends only on y and (1.7) is a special case of the "geometrico-acoustical" solution (2.7) (taking account of the different choice of direction of propagation of the waves).

2. Geometric acoustics of nonstationary media. We shall now assume that the variation of the parameters of the medium correspond to some arbitrary motion. Here in the general case a gravity field $g(x, t)$ is also present and the medium may be inside a tube of variable cross-sectional area $\Delta(x, t)$ (then x is measured along the axis of the tube).

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* The fact that the amplitude of the wave can increase sharply in a region of compression because of the decrease in the region of rarefaction is of interest by itself.

This last assumption allows us to generalize the solution to the three-dimensional case (see below). It also permits consideration of a problem which is important in a number of applications, that of longitudinal motion of a conducting plasma in a strong magnetic field **H** (if $p \ll p_m = H^2/8\pi$ the motion takes place along the tubes of force of **H**, owing to confinement). Under the assumptions which have been made, the right-hand sides of the first and second equations of (1.1) are g and $-(\partial\Delta/\partial t + u\partial\Delta/\partial x)\rho/\Delta$, respectively.

Let us consider the propagation of short waves whose characteristic dimensions in space, λ, and in time, τ, are small compared to the measures of variation of the parameters of the medium L and T, respectively. We again use the transformation (1.2) where u_0 and ρ_0, the parameters of the undisturbed medium (undisturbed relative to the wave under study), are now slowly varying functions of x and t, and the integral in (1.3) is taken for constant S. The basic equations then give the following for φ and ψ:

$$c\frac{\partial\psi}{\partial x} - \varphi = g + \frac{cc_0}{\rho_0}\frac{\partial\rho_0}{\partial x} - c\frac{\partial u_0}{\partial x} + \frac{\partial S}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial S} - c\int\limits_{\rho_0}^{\rho}\frac{1}{\rho}\frac{\partial c}{\partial S}\,d\rho\right)$$
$$\frac{D\psi}{Dt} - \varphi = \frac{c_0}{\rho_0}\frac{D\rho_0}{Dt} - \frac{Du_0}{Dt} + \frac{c}{\Delta}\frac{D\Delta}{Dt} \qquad \left(c_0 = c(\rho_0, S),\ \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) \tag{2.1}$$

Equations (2.1) are the exact consequence of the basic system (1.1). In the present section we shall restrict ourselves to the linear problem, with $\rho = \rho_0 + \rho'$, $u = u_0 + u'$, etc. where ρ' and u' are small. We also represent φ in the form $\varphi_0 + \varphi'$, where φ_0 is a slowly varying function equal to $(du_0/dt)_+$ in accordance with (1.2). Eliminating ψ from (2.1) by differentiation (direct determination of ψ is not necessary in the present approximation), we obtain

$$\left(\frac{d\varphi'}{dt}\right)_- = \frac{\partial\varphi'}{\partial t} + (u_0 - c_0)\frac{\partial\varphi'}{\partial x} = \frac{\partial u'}{\partial x}\left\{\frac{\partial\rho_0 c_0}{\partial\rho_0}\left(\frac{c_0}{\rho_0}\frac{\partial\rho_0}{\partial x} - \frac{\partial u_0}{\partial x}\right) + \right.$$
$$\left. + \frac{c_0^2}{\Delta}\frac{\partial\Delta}{\partial x} + \frac{\rho_0}{\Delta}\frac{\partial c_0}{\partial\rho_0}\frac{D\Delta}{Dt} + c_0\frac{\partial c_0}{\partial S_0}\frac{\partial S_0}{\partial x}\right\} = f(x, t)\frac{\partial u'}{\partial x} \tag{2.2}$$

Here the nonlinear terms are omitted and it is considered that the perturbation of the entropy is of the second order relative to u'. The slowly varying function in the braces is denoted by f.

If φ' is known, the unknown function u' is determined by the linearized equation (1.2)

$$\frac{\partial u'}{\partial t} + v_0\frac{\partial u'}{\partial x} = \varphi' - \frac{1}{c_0}\frac{\partial c_0\rho_0}{\partial\rho_0}\frac{\partial u_0}{\partial x} \tag{2.3}$$

Equation (2.2) can be integrated by finding φ' in the form $a(x,t)u'$, where a is a slowly varying function. Then, to within second-order quantities

$$\left(\frac{d\varphi'}{dt}\right)_- = a\left(\frac{du'}{dt}\right)_- = -2ac_0\frac{\partial u'}{\partial x}$$

(the last relation is clear from (2.3) whose right-hand side is obviously small for slow variations of the parameters). Therefore

$$\varphi' = -\frac{f(x,t)}{2c_0}u' \tag{2.4}$$

Taking account of (2.4), we can also integrate equation (2.3). If $\delta(x,t)$ is the integral of the equation $dx/dt = v_0$ which defines the c_+-characteristic of the undisturbed system, the general integral of (2.3) has the form

$$\delta(x,t) = t - \int^{c+}\frac{dx}{v_0(x,t)} = F\left(\frac{u'}{q}\right), \qquad q = \left(\frac{c_0}{\Delta\rho_0 v_0^2}\right)^{1/2}\exp\left(\int^{c+}\frac{1}{v_0}\frac{\partial v_0}{\partial t}dt\right) \tag{2.5}$$

(q is a slowly varying factor which determines the variation of u' on the characteristic). For steady motions or no motion of the medium (2.5) coincides with known solutions in the approximation of geometric acoustics. It should be noted that the solution (2.5) is also valid in the three-dimensional case if Δ is identified with the cross-sectional area of an elementary phase tube formed by the normals $\mathbf{n}$ to the wave front (which differs from a ray tube in a moving medium). This follows from the fact that $\mathbf{n}$ and u' are parallel and confirms previously obtained results in the stationary case [2,7].

By the use of (2.5) it is possible to determine the time variation of the duration τ and of the energy E of a wave (pulse). (In a stationary medium these quantities are obviously constant.) The quantity τ is the time interval between the two characteristics (2.5) which bound the wave. Considering that the quantity v_0 changes only slightly in time τ, it is easy to show that

$$\tau + \int^{c+}\frac{\tau}{v_0}\frac{\partial v_0}{\partial t}dt = \tau_1, \quad \text{or} \quad \tau = \tau_1\exp\left(-\int^{c+}\frac{1}{v_0}\frac{\partial v_0}{\partial t}dt\right) \tag{2.6}$$

where τ_1 is the initial value of τ.

Since the energy density of the wave equals [7]

$$e = \frac{v_0}{c_0}\rho_0 u'^2 \tag{2.7}$$

the total energy in the wave form is

$$E = e\tau v_0 = E_1 \exp\left(\int^{c+} \frac{1}{v_0}\frac{\partial v_0}{\partial t}\,dt\right) \tag{2.8}$$

and, therefore

$$E\tau = \mathrm{const} \tag{2.9}$$

Thus, as v_0 increases with time, the duration of the wave form decreases and the total energy becomes larger (this also refers to any period of a quasiharmonic wave). The quantity $E\tau$ is an adiabatic invariant in the same sense as for systems with finite numbers of degrees of freedom.

3. Nonlinear motions. The results of the previous section can be generalized easily to the case of slight nonlinearity where u' and ρ' are finite, so that the distortions of the wave caused by nonlinear effects and by the variation of the parameters of the system develop in comparable intervals.

If small quantities of the order of the square of u' are considered, expression (2.4) for φ' remains correct inasmuch as the right-hand side of (2.3) is already small by virtue of the slow variation of the parameters. However, on the left-hand side of (2.3) we must make the replacement

$$v_0 \to v \approx v_0 + \frac{1}{c_0}\frac{\partial \rho_0 c_0}{\partial \rho_0}\,u'$$

We then obtain, instead of (2.7)

$$\delta\,(x, t) + \frac{u'}{q}\int^{c+} \frac{1}{c_0}\frac{\partial \rho_0 c_0}{\partial \rho_0}\frac{\partial \delta}{\partial t}\,q\,dt = F\left(\frac{u'}{q}\right) \tag{3.1}$$

For the stationary case (here $\partial\delta/\partial t = 1$) the solution (3.1) also coincides with results obtained previously (taking account of the comment made above on the three-dimensional problem) [5].

In particular, formula (3.1) permits us to examine problems connected with the origination and development of shock waves. The values x_*, t_* and u_*' which correspond to the formation of a discontinuity are determined, as usual, by the conditions $\partial t/\partial u' = 0$, $\partial^2 t/\partial u'^2 = 0$.* We then find from (3.1)

* If the discontinuity begins at the boundary with the undisturbed medium then, instead of the second condition, u' should be set equal to zero [3].

$$\int^{c_+} \frac{q}{v_0 c_0} \frac{\partial c_0 \rho_0}{\partial \rho_0} \frac{\partial \delta}{\partial t} dt = \frac{dF}{d\alpha}, \qquad \frac{d^2 F}{d\alpha^2} = 0 \qquad \left(\alpha = \frac{u'}{q}\right) \tag{3.2}$$

where α is a quantity which is conserved on the c_+-characteristic.

We note that if the integrand in (3.2) decreases so rapidly that the integral remains finite for $t_* \to \infty$, a discontinuity may not occur at all even in a compression wave. It is not difficult to see that this peculiarity is not restricted to the nonstationary problem. For instance, let a sinusoidal perturbation begin in a steadily moving gas; i.e. $u'(0, t) = u^m \sin \omega t$ (it is obviously sufficient to examine one period of the sine wave). It then follows from (3.2) that

$$\frac{\gamma + 1}{2} u^m \omega \left(\frac{\Delta_0 \rho_{00} v_{00}^2}{c_{00}}\right)^{1/2} \int_0^{x_*} \frac{dx}{v_0^3 (\Delta \rho_0 / c_0)^{1/2}} = 1, \qquad u_*' = 0 \tag{3.3}$$

where γ is the adiabatic exponent, $\rho_{00} = \rho_0(0)$, etc. For a plane wave propagating in a homogeneous gravity field in an isothermal atmosphere at rest, we have as a result

$$x_* = X_* \frac{\gamma(\gamma + 1) \omega u^m}{g} \ln \left[\left(1 - \frac{g}{\gamma(\gamma + 1) \omega u^m}\right)^{-1}\right] \qquad \left(X_* = \frac{2c_0^2}{(\gamma + 1) \omega u^m}\right) \tag{3.4}$$

Here X_* is the distance corresponding to the formation of a discontinuity in a homogeneous medium (for $g = 0$). It follows from (3.3) that for $g > 0$ (propagation toward the center of attraction) $x_* > X_*$; i.e. nonhomogeneity of the gas hinders formation of the discontinuity. If $\gamma(\gamma + 1)\omega u^m > g$ the discontinuity does not occur at all.* On the other hand, for $g < 0$ (propagation away from the center of attraction) $x_* < X_*$ and the discontinuity can begin considerably more rapidly than in a homogeneous medium.** The cause of this is the variation of u' on the characteristics and the associated differences of velocity of propagation of the various points on the profile of the wave.

As is known, after the formation of a discontinuity the wave is simple up to third-order quantities away from the discontinuity. If

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* An analogous conclusion holds for converging spherical waves in a central gravity field ($\Delta \sim x^2$, $g \sim x^{-2}$), even without considering reflections which occur near the center (for $x \leqslant \lambda$).

** This last situation is important for a number of astrophysical applications.

there is no disturbance in front of the discontinuity, then by differentiating (2.4) with respect to α along the shock-wave trajectory we obtain an equation which determines the magnitude of the shock α_s

$$\int^{c+} \frac{q}{v_0 c_0} \frac{\partial \rho_0 c_0}{\partial \rho_0} \frac{\partial \delta}{\partial t} = \frac{2}{\alpha_s^2} \int_{\alpha_0}^{\alpha_s} \alpha \frac{dF}{d\alpha} d\alpha \tag{3.5}$$

In the case of a wave form of finite duration the integral

$$B(\alpha_s) = \int_{\alpha_0}^{\alpha_s} \alpha \, dF(\alpha)$$

on the right-hand side of (3.5) cannot be larger than the finite quantity $B(0)$. If the integral on the left of (3.5) diverges for $t \to \infty$ (as in a homogeneous medium) then α decreases (to zero) without a finite bound and $B \to B(0)$. Here the profile of the wave form behind the discontinuity approaches a linear shape at great distances. There is, however, also another possibility, when the above-mentioned integral remains finite as $t \to \infty$ (compare (3.1)). In this case α approaches a finite limit and B does not reach $B(0)$. Then the profile of the wave will not be linear as $t \to \infty$.

As has already been pointed out, the energy of a wave in a nonstationary medium can increase as a result of the energy of motion of the "undisturbed" medium. Under certain conditions this increase takes place more rapidly than the dissipation of energy in a shock wave (at least in finite intervals), so that the total energy in the wave form may increase even in the presence of a discontinuity.

In conclusion, we remark that the method which has been set forth allows us to consider analogous problems relating to the propagation of magnetohydrodynamic and electromagnetic waves [8] in media with variable properties. In some cases there is a direct mathematical analogy between the equations for acoustic waves and those for electromagnetic waves in such media [9].

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